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# Characterizing Energy-Related Controllability of Complex Networks Via Cartesian Product

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***Abstract*—This work investigates energy-related controllability of composite complex networks constructed by factor networks via Cartesian graph product. Particularly, the considered factor networks are leader-follower signed networks with neighbor-based Laplacian dynamics, adopting positive and negative edges to capture cooperative and competitive interactions among network units. Instead of considering classical controllability, energy-related metrics of composite networks capturing system performance, such as average controllability and volumetric control energy, are characterized through their corresponding factor networks. It is revealed that the eigenvalues and eigenvector matrices of factor systems can be used to characterize energy-related controllability of composite networks.**

## I. INTRODUCTION

Complex networks can effectively model a variety of natural and man-made systems. Brain networks, metabolic networks, and biological immune networks are typical applications of the complex networks in nature, while multi-agent systems, power networks, and transportation networks are instances of modern engineering systems. Owing to tremendous application potential, growing research has been devoted to investigating the structural and functional properties of complex networks. From the viewpoint of advancing design and control of complex networks, properties that are of particular interest to us are the controllability and energy-related performance of complex networks.

This work focuses on composite complex networks, i.e., networked systems composed of factor systems. Such networked systems are widely seen in biological networks and power system modeling [1] [2]. It has been observed that many complex systems can be constructed and analyzed through simple subsystems (i.e., factors), where core properties of factors are preserved in the composite system. For example, controllability and observability of factor systems are preserved under series-parallel connections [3]. Stability of composite feedback systems can be analyzed based on its factor systems via small-gain theorem and composite Lyapunov functions [4]. Recently, graph products have been explored to construct and reveal structural and functional relationships between factor systems and the associated composite system. Cartesian product is one of the graph products used to obtain a composite network from simple factor networks. In [5]–[8], classical controllability and observability of a composite network were characterized based on its factor networks. In [9], the verification and prediction of the structural balance of signed networks were studied via Cartesian product. In a recent work [10], generalized graph product, including Cartesian, direct, and strong product, was utilized to reveal spectral and controllability properties of composite systems.

Different from previous works [11] and [12], the presented work characterizes the energy-related controllability of composite complex networks. In particular, we consider a class of composite networks constructed from simple factor networks via Cartesian product. For each factor network, the network units are classified as either leaders or followers interacting via neighbor-based Laplacian feedback. The factor network allows positive and negative edges to capture cooperative and competitive interactions among network units. Due to the graph product, the resulting composite network is a signed leader-follower network. The goal is to investigate the energy-related controllability of composite networks. Since direct analysis over a large-scale composite network can be challenging, this work focuses on leveraging graph product approaches to explore how the energy-related controllability of the composite network can be inferred from its factor systems.

The contributions of this work are multi-fold. First, this work characterizes energy-related controllability of complex composite networks via graph product approaches. A crucial benefit of using graph product is that the global properties, such as average controllability and volumetric control energy, of the composite networks can be inferred from its local factor graphs. This work is closely related to [5], where classical network controllability was investigated via Cartesian graph product to provide conditions under which the network is controllable via external inputs. Different from [5], the presented work focuses on characterizing energy-related measures of network controllability. It is revealed that the eigenvalues and eigenvector matrices of factor systems can be used to characterize energy-related controllability of composite networks. In addition, the presented work considers signed networks with Laplacian dynamics. Signed networks can model a large class of networks with cooperative and competitive interactions among network units, such as social networks and resilient networks [13]. Therefore, the developed energy-related characterizations are not only applicable to unsigned networks (i.e., cooperative networks with non-negative edge weights), but also to competitive

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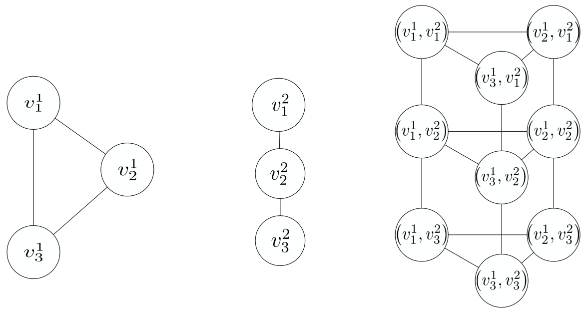


Figure 1. Factor graphs (a) and (b), and their product graph (c) $\mathcal{G}_1 \square \mathcal{G}_2$.

networks with possible antagonistic interactions.

## II. Preliminaries

### A. Cartesian Product

Complex networks can be synthesized from a set of smaller size factor graphs via graph product [14]. In this section, Cartesian graph product is introduced, which will be used as a main tool to characterize energy-related controllability of complex networks in the subsequent analysis. Consider two undirected graphs $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1, \mathcal{A}_1)$ and $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2, \mathcal{A}_2)$. In $\mathcal{G}_1$, $\mathcal{V}_1 = \{v_1^1, \ldots v_n^1\}$ represents the set of $n$ nodes, $\mathcal{E}_1 = \mathcal{V}_1 \times \mathcal{V}_1$ represents the edge set, and $\mathcal{A}_1 = \left[a_{ij}^1\right] \in \mathbb{R}^{n \times n}$ is the adjacency matrix with $a_{ij}^1 \neq 0$ if $\left(v_i^1, v_j^1\right) \in \mathcal{E}_1$ and $a_{ij}^1 = 0$ otherwise. The graph $\mathcal{G}_2$ is defined similarly with $\mathcal{V}_2 = \{v_1^2, \ldots v_m^2\}$, $\mathcal{E}_2 = \mathcal{V}_2 \times \mathcal{V}_2$, and $\mathcal{A}_2 = \left[a_{ij}^2\right] \in \mathbb{R}^{m \times m}$. Denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}) = \mathcal{G}_1 \square \mathcal{G}_2$ the composite graph constructed by the Cartesian product of two factor graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, where $\square$ denotes the Cartesian product. The Cartesian product of $\mathcal{G}_1$ and $\mathcal{G}_2$ satisfies the condition that an edge $\left(\left(v_i^1, v_p^2\right), \left(v_j^1, v_q^2\right)\right) \in \mathcal{E}$ exists if and only if either $\left(v_i^1, v_j^1\right) \in \mathcal{E}_1$ and $v_p^2 = v_q^2$, or $\left(v_p^2, v_q^2\right) \in \mathcal{E}_2$ and $v_i^1 = v_j^1$. The non-zero entry $a_{((i,p),(j,q))}$ in $\mathcal{A} \in \mathbb{R}^{mn \times mn}$ corresponding to the edge $\left(\left(v_i^1, v_p^2\right), \left(v_j^1, v_q^2\right)\right)$ in $\mathcal{G}$ is defined as

$$a_{((i,p),(j,q))} = \delta_{pq} a_{ij}^1 + \delta_{ij} a_{pq}^2,$$

where $\delta_{uv} = 1$ if $u = v$ and $\delta_{uv} = 0$ otherwise, and $a_{ij}^1$ and $a_{pq}^2$ are entries in $\mathcal{A}_1$ and $\mathcal{A}_2$ corresponding to $\left(v_i^1, v_j^1\right)$ and $\left(v_p^2, v_q^2\right)$, respectively.

An example of Cartesian product is illustrated in Fig. 1. Note that the Cartesian product is commutative and associative, i.e., $\mathcal{G}_1 \square \mathcal{G}_2$ and $\mathcal{G}_2 \square \mathcal{G}_1$ are isomorphic, and $(\mathcal{G}_1 \square \mathcal{G}_2) \square \mathcal{G}_3$ and $\mathcal{G}_1 \square (\mathcal{G}_2 \square \mathcal{G}_3)$ are isomorphic for any factor graphs $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$.

### B. Kronecker Product and Sum

Consider two matrices $A_1 \in \mathbb{R}^{n \times n}$ and $A_2 \in \mathbb{R}^{m \times m}$. The Kronecker product of $A_1$ and $A_2$ is denoted by $A_1 \otimes A_2 \in \mathbb{R}^{nm \times nm}$. Further, the Kronecker sum of $A_1$ and $A_2$ is defined as $A_1 \oplus A_2 = A_1 \otimes I_m + I_n \otimes A_2$, where $I_u$ is a $u \times u$ identity matrix. The Kronecker product has the following properties: $(A_1 \otimes A_2)(A_3 \otimes A_4) = (A_1 A_3) \otimes (A_2 A_4)$ and $e^{A_1 \oplus A_2} = e^{A_1} \otimes e^{A_2}$. The spectrum of matrix $A$ is denoted as $\operatorname{eig}(A)$, i.e., the set of eigenvalues of $A$. More detailed treatment of Kronecker product and sum can be found in [15].

## III. Problem Formulation

### A. Leader-Follower Signed Factor Network

Consider a complex network represented by an undirected signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where the node set $\mathcal{V} = \{v_1, \ldots, v_n\}$ and the edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ represent the network units and their interactions, respectively. The interactions are captured by the adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n \times n}$, where $a_{ij} \neq 0$ if $(v_i, v_j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. No self-loop is considered, i.e., $a_{ii} = 0$ $\forall i = 1, \ldots, n$. Different from many existing results considering exclusively non-negative $a_{ij}$ (i.e., an unsigned graph), this work adopts $a_{ij} \in \{\pm 1\}$, allowing the signed edge to capture both cooperative and competitive interactions between network units. Let $d_i = \sum_{j \in \mathcal{N}_i} |a_{ij}|$, where $\mathcal{N}_i = \{v_j | (v_i, v_j) \in \mathcal{E}\}$ denotes the neighbor set of $v_i$ and $|a_{ij}|$ denotes the absolute value of $a_{ij}$. The graph Laplacian of $\mathcal{G}$ is defined as $\mathcal{L}(\mathcal{G}) \triangleq \mathcal{D} - \mathcal{A}$, where the in-degree matrix $\mathcal{D} \triangleq \operatorname{diag}\{d_1, \ldots, d_n\}$ is a diagonal matrix. Since $\mathcal{G}$ is undirected, the graph Laplacian $\mathcal{L}(\mathcal{G})$ is symmetric. Note that, when considering signed graphs, the graph Laplacian $\mathcal{L}(\mathcal{G})$ may have negative off-diagonal entries and its row/column sums are not necessarily zero, which indicates that zero is no longer a default eigenvalue as in the case of unsigned graphs.

Let $x(t) \in \mathbb{R}^n$ denote the stacked system states, where the $i$th entry represents the state of node $v_i$. Consider a set $\mathcal{K} = \{v_{l_1}, \ldots, v_{l_m}\} \subseteq \mathcal{V}$ of nodes endowed with external control inputs (i.e., the leaders), where $l_i$, $i = 1, \ldots m$, indicates the leader's index. Suppose the system states evolve over the signed graph $\mathcal{G}$ according to the following Laplacian dynamics,

$$\dot{x}(t) = -\mathcal{L}(\mathcal{G}) x(t) + Bu(t), \tag{1}$$

where the graph Laplacian $\mathcal{L}(\mathcal{G})$ indicates that each node updates its state by taking into account the states of its neighboring nodes, $u(t) \in \mathbb{R}^m$ is the external input, and $B = \left[\begin{array}{ccc} e_{l_1} & \cdots & e_{l_m} \end{array}\right] \in \mathbb{R}^{n \times m}$ is the input matrix with basis vector[1] $e_i \in \mathbb{R}^n$, $i = l_1, \ldots, l_m$, indicating the leaders are endowed with external controls. For notational simplicity, $(\mathcal{L}, B)$ will be used throughout this work to represent the dynamics in (1).

### B. Control Energy Metrics

The leader-follower system $(\mathcal{L}, B)$ can be controllable with appropriate selection of leaders in $\mathcal{G}$ (i.e., an appropriate design of $B$). Most existing results focus on designing $B$ to ensure classical network controllability, such that the system state can be driven from an initial state $x(0) \in \mathbb{R}^n$ to any

[1] A basis vector $e_i \in \mathbb{R}^n$ has zero entries except for the $i$th entry being one.

target state $x_t \in \mathbb{R}^n$ by an external input $u(t)$. The total energy required in network control over a time interval $[0, t]$ can be quantified as

$$E(t) = \int_0^t \|u(\tau)\|^2 \, d\tau. \tag{2}$$

Assuming the initial state $x(0) = 0$ and the optimal control $u(t)$ in [16], the minimum control energy required to drive the system $(\mathcal{L}, B)$ from $x(0)$ to $x_f$ is

$$E(t) = x_f^T \mathcal{W}^{-1}(t) x_f, \tag{3}$$

where

$$\mathcal{W}(t) = \int_0^t e^{-\mathcal{L}\tau} B B^T e^{-\mathcal{L}^T \tau} d\tau$$

is the controllability Gramian. In this work we focus on the infinite horizon case, i.e., $t \to \infty$, due to the consideration of asymptotic or exponential convergence/stability of dynamic systems.

Since the controllability Gramian provides an energy-related measure of network control, various metrics have been developed based on $\mathcal{W}$. Typical control energy metrics include the worst case control energy, average controllability $\operatorname{tr}(\mathcal{W})$, volumetric control energy $\log\det(\mathcal{W})$, and average control energy $\operatorname{tr}\left(\mathcal{W}^{-1}\right)$ [17]. Based on the controllability Gramian, the above metrics provide energy-related measures of network controllability. Such measures are referred to as energy-related controllability in this work and the subsequent effort will focus on characterizing the energy-related controllability of composite complex networks, mainly on average controllability and volumetric control energy.

### C. Composite Complex System

This section shows how the leader-follower factor networks can be synthesized to represent a composite complex system. Consider a set of $s$ leader-follower systems $(\mathcal{L}_i, B_i)$ evolving over factor graphs $\mathcal{G}_i$, $i = 1, \ldots, s$, respectively. According to (1), the dynamics of the $i$th factor system is

$$\dot{x}_i(t) = -\mathcal{L}_i x_i(t) + B_i u_i(t), \tag{4}$$

where $x_i(t) \in \mathbb{R}^{n_i}$ is the system state, $\mathcal{L}_i = \mathcal{L}(\mathcal{G}_i) \in \mathbb{R}^{n_i \times n_i}$ represents the associated graph Laplacian of $\mathcal{G}_i$, $B_i \in \mathbb{R}^{n_i \times m_i}$ represents the input matrix encoding the leaders, and $u_i(t) \in \mathbb{R}^{m_i}$ is the external input.

Based on the introduced graph product in Sect. II-A, let $\mathcal{G} = \prod_{\square} \mathcal{G}_i$ be the composite graph constructed from the factor graphs $\mathcal{G}_i$ via Cartesian product, $i = 1, \ldots, s$. Provided that each $\mathcal{G}_i$ evolves according to (4), the composite dynamics $(\mathcal{L}, B)$ over $\mathcal{G}$ can be written as

$$\begin{aligned} \dot{x}(t) &= -\mathcal{L}\left(\prod_{\square} \mathcal{G}_i\right) x(t) + \left(\prod_{\otimes} B_i\right) u(t) \\ &= -\mathcal{L}(\mathcal{G}) x(t) + B u(t), \end{aligned} \tag{5}$$

where $x(t) \in \mathbb{R}^{\prod_{i=1}^s n_i}$ is the system state, $\mathcal{L} \in \mathbb{R}^{\prod_{i=1}^s n_i \times \prod_{i=1}^s n_i}$ is the graph Laplacian of $\mathcal{G}$, $B \in \mathbb{R}^{\prod_{i=1}^s n_i \times \prod_{i=1}^s m_i}$ represents the input matrix, and $u(t) \in \mathbb{R}^{\prod_{i=1}^s m_i}$ is the external input. The composite system $(\mathcal{L}, B)$ evolves over the composite graph $\mathcal{G}$ with $B$ encoding the leaders inherited from $\mathcal{G}_i$, $i = 1, \ldots, s$. Specifically, given a leader set $\mathcal{K}_i$ in $\mathcal{G}_i$, if $B_i = B(\mathcal{K}_i)$ denotes the input matrix generated from $\mathcal{K}_i$, then the input matrix $B$ of $\mathcal{G}$ can be written as $B = \prod_{\otimes} B(\mathcal{K}_i) = B(\mathcal{K})$, where $\mathcal{K} = \prod_{\times} \mathcal{K}_i$ represents the set of leaders in $\mathcal{G}$ [5].

Direct analysis of the energy-related controllability of a composite network can be challenging, especially when the composite network is of large size. Hence, the objective is to characterize the energy-related controllability of the composite system by inferring from its factor systems, taking advantage of the smaller size of the factor systems.

The subsequent development focuses on a composite system $(\mathcal{L}(\mathcal{G}), B)$ constructed by the Cartesian product of two factor systems $(\mathcal{L}(\mathcal{G}_1), B_1)$ and $(\mathcal{L}(\mathcal{G}_2), B_2)$, where $\mathcal{G}_1$ and $\mathcal{G}_2$ are undirected signed graphs with $n$ and $m$ nodes, respectively. The case of two factor systems is adopted for the simplicity of presentation and is not restrictive, since a general composite system with many factor systems can be realized via sequential composition. In addition, the assumption that a complex network can be decomposed into factor system is also a mild assumption, since any graph has a prime factor decomposition [5].

## IV. Energy-Related Controllability of Cartesian Product Graph

This section focuses on the characterizations of energy-related controllability of composite networks that are constructed via Cartesian product. To better explain the idea, we start from the case that each factor system contains a single leader. Specifically, consider two factor systems $(\mathcal{L}(\mathcal{G}_1), b_1)$ and $(\mathcal{L}(\mathcal{G}_2), b_2)$ with $b_1 = e_{l_1}$ and $b_2 = e_{l_2}$, where the basis vectors $e_{l_1}$ and $e_{l_2}$ indicate that $v_{l_1}$ and $v_{l_2}$ are the leaders in $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively, and $l_1 \in \{1, \ldots, n\}$ and $l_2 \in \{1, \ldots, m\}$. By Cartesian product, the composite graph is constructed as $\mathcal{G} = \mathcal{G}_1 \square \mathcal{G}_2$. Since graph Laplacians belong to the family of symmetry preserving representations[2], as proved in [5], $\mathcal{L}(\mathcal{G}_1 \square \mathcal{G}_2) = \mathcal{L}(\mathcal{G}_1) \oplus \mathcal{L}(\mathcal{G}_2)$. Hence, the dynamics of the composite system $(\mathcal{L}, b_l)$ can be written as

$$\begin{aligned} \dot{x}(t) &= -\mathcal{L} x(t) + b_l u(t) \\ &= -(\mathcal{L}_1 \oplus \mathcal{L}_2) x(t) + (e_{l_1} \otimes e_{l_2}) u(t), \end{aligned} \tag{6}$$

where the input vector $b_l = e_{l_1} \otimes e_{l_2} \in \mathbb{R}^{mn}$ determines the leader $v_l$ in $\mathcal{G}$. Based on (6), average controllability and volumetric control energy are explored in the following sections to characterize the energy-related controllability of $(\mathcal{L}, b_l)$ based on its factor systems $(\mathcal{L}_1, b_1)$ and $(\mathcal{L}_2, b_2)$.

Different from unsigned graphs whose graph Laplacian is positive semi-definite by default, when considering signed networks, the graph Laplacian $\mathcal{L}$ can be either positive semi-definite (i.e., $\mathcal{G}$ is structurally balanced) or positive definite (i.e., $\mathcal{G}$ is structurally unbalanced) [13]. The subsequent

[2]A matrix $\mathcal{L}(\mathcal{G})$ is symmetry preserving if, for all permutation $\sigma \in \operatorname{Aut}(\mathcal{G})$, with the corresponding permutation matrix $J$, $\mathcal{L}(\mathcal{G}) J = J \mathcal{L}(\mathcal{G})$.

development will focus on the cases that $\mathcal{G}$ is structurally unbalanced, i.e., $\mathrm{eig}(\mathcal{L})$ contains only positive eigenvalues. If $\mathcal{G}$ is structurally balanced, then, as shown in our recent work [18], a structurally balanced graph can be converted to an unsigned graph under gauge transformation [13], where many existing energy-related characterizations (cf. [19] and [17]) can be immediately applied. In addition, as shown in [20]–[22], the reduced graph Laplacian can be used, where the row and column associated with the zero eigenvalue are removed from the graph Laplacian, so that a structurally balanced graph can be treated as a structurally unbalanced graph via reduced Laplacian matrix when deriving Gramian-based energy metrics. Therefore, we mainly focus on the energy-related characterizations of structurally unbalanced graphs.

### A. Characterizations of Average Controllability

Before characterizing the average controllability of the composite system $(\mathcal{L}, b_l)$ in (6), the following lemma from [23] is introduced.

**Lemma 1.** *Consider two factor graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ with $n$ and $m$ nodes, respectively. Given the Cartesian product graph $\mathcal{G} = \mathcal{G}_1 \square \mathcal{G}_2$, the graph Laplacian $\mathcal{L}(\mathcal{G})$ takes the form of $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2 = \mathcal{L}_1 \otimes I_m + I_n \otimes \mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are the graph Laplacian of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. The eigenvalues $\lambda_k$ and eigenvectors $u_k$ of $\mathcal{L}$ are defined as $\lambda_k = \mu_i + \eta_j$ and $u_k = \vartheta_i \otimes w_j$ for $k = 1, \dots mn$, where $(\mu_i, \vartheta_i)$, $i = 1, \dots, n$, and $(\eta_j, w_j)$, $j = 1, \dots m$, represent the eigenpairs of $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively.*

Lemma 1 shows how the eigenpairs of $\mathcal{L}$ can be constructed from the eigenpairs of $\mathcal{L}_1$ and $\mathcal{L}_2$. Based on Lemma 1, the following theorem characterizes the average controllability of $(\mathcal{L}, b_l)$. To better explain the idea, let $\vartheta = [\vartheta_1 \cdots \vartheta_n]$ and $w = [w_1 \cdots w_m]$ denote the orthogonal eigenvector matrices of $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively.

**Theorem 1.** *Provided two factor systems $(\mathcal{L}(\mathcal{G}_1), b_1)$ and $(\mathcal{L}(\mathcal{G}_2), b_2)$, the average controllability of the composite system $(\mathcal{L}(\mathcal{G}_1 \square \mathcal{G}_2), b_l)$ in (6) can be characterized as*

$$\mathrm{tr}(\mathcal{W}) = \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{1}{2(\mu_i + \eta_j)} \vartheta_{l_1,i}^2 w_{l_2,j}^2, \tag{7}$$

*where $\mathcal{W}$ is the controllability Gramian of $(\mathcal{L}, b_l)$, and $\mu_i$ for $i = 1, \dots, n$ and $\eta_j$ for $j = 1, \dots m$ being the spectra of $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, and $\vartheta_{l_1,:}$ and $w_{l_2,:}$ being $l_1 th$ and $l_2 th$ row of the eigenvector matrices $\vartheta$ and $w$, respectively, determined by the leaders $l_1$ and $l_2$.*

*Proof:* Consider the composite system $(\mathcal{L}, b_l)$. Since $\mathcal{L}$ is symmetric, it can be written as $\mathcal{L} = U \Lambda U^T \in \mathbb{R}^{mn \times mn}$, where $\Lambda = \mathrm{diag}\{\lambda_1, \dots, \lambda_{mn}\} \in \mathbb{R}^{mn \times mn}$ is a diagonal matrix containing the eigenvalues of $\mathcal{L}$ and $U = \begin{bmatrix} u_1 & \cdots & u_{mn} \end{bmatrix} \in \mathbb{R}^{mn \times mn}$ is the orthogonal eigenvector matrix of $\mathcal{L}$. Based on (6) and using the fact that $e^{-\mathcal{L}\tau} = e^{-U \Lambda U^T \tau} = U e^{-\Lambda \tau} U^T$, the controllability Gramian can be written as

$$\mathcal{W} = \int_0^\infty e^{-\mathcal{L}\tau} b_l b_l^T e^{-\mathcal{L}\tau} d\tau = U \varGamma U^T, \tag{8}$$

where

$$\varGamma = \int_0^\infty e^{-\Lambda\tau} U^T b_l b_l^T U e^{-\Lambda\tau} d\tau. \tag{9}$$

From (9), the $ij$th entry of $\varGamma$ is

$$\varGamma_{i,j} = \int_0^\infty e^{-\lambda_i \tau - \lambda_j \tau} u_{l,i} u_{l,j} d\tau = \frac{1}{\lambda_i + \lambda_j} u_{l,i} u_{l,j}, \tag{10}$$

where $u_{l,i}$ and $u_{l,j}$ are the $li$th and $lj$th entries of $U$, respectively.

Since the trace is invariant under cyclic permutations, from (8), the average controllability of $(\mathcal{L}, b_l)$ is

$$\mathrm{tr}(\mathcal{W}) = \mathrm{tr}\left(U \varGamma U^T\right) = \mathrm{tr}\left(\varGamma U^T U\right). \tag{11}$$

Substituting (10) into (11) and using Lemma 1,

$$\begin{aligned} \mathrm{tr}(\mathcal{W}) &= \mathrm{tr}(\varGamma) \\ &= \sum_{k=1}^{mn} \frac{1}{2\lambda_k} u_{l,k}^2 \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{1}{2(\mu_i + \eta_j)} \vartheta_{l_1,i}^2 w_{l_2,j}^2, \end{aligned} \tag{12}$$

where $\vartheta_{l_1,:}$ and $w_{l_2,:}$ are the $l_1 th$ and $l_2 th$ row of the eigenvector matrices $\vartheta$ and $w$, respectively, corresponding to the leader node $v_{l_1}$ and $v_{l_2}$. ■

A key observation from Theorem 1 is that, for a Cartesian product composite graph with a single leader, the average controllability can be inferred from the eigenvalues of the factor graph Laplacian and the associated rows of the eigenvector matrices corresponding to the leaders. In other words, the average controllability of a complex large-scale network can be analyzed based on its relatively simple factor systems. To further clarity, suppose a composite graph Laplacian $\mathcal{L}$ has a dimension of $mn \times mn$ with its factor graph Laplacians $\mathcal{L}_1$ and $\mathcal{L}_2$ of size $n \times n$ and $m \times m$. Instead of analyzing $\mathcal{L}$ of large size $mn$, Theorem 1 provides a practical means to characterize $\mathrm{tr}(\mathcal{W})$ only based on the eigenvalues and eigenvectors of $\mathcal{L}_1$ and $\mathcal{L}_2$. Thus, offering a bottom up approach to reveal the properties of a global system from its local systems.

Based on the single leader case in Theorem 1, the following theorem considers multi-leader cases.

**Theorem 2.** *Consider two factor systems $(\mathcal{L}(\mathcal{G}_1), B_1)$ and $(\mathcal{L}(\mathcal{G}_2), B_2)$, where $\mathcal{G}_1$ has $n$ nodes with $p$ leaders and $\mathcal{G}_2$ has $m$ nodes with $q$ leaders, i.e., $B_1 = \begin{bmatrix} e_{l_1^1} & \cdots & e_{l_p^1} \end{bmatrix} \in \mathbb{R}^{n \times p}$ and $B_2 = \begin{bmatrix} e_{l_1^2} & \cdots & e_{l_q^2} \end{bmatrix} \in \mathbb{R}^{m \times q}$ where the basis vector $e_i$, $i \in \{l_1^1, \dots, l_p^1\}$, and $e_j$, $j \in \{l_1^2, \dots, l_q^2\}$, indicate the leader nodes $v_i$ and $v_j$ in $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. The average controllability of the composite system*

$(\mathcal{L}(\mathcal{G}_1 \square \mathcal{G}_2), B)$ *can be characterized as in (15).*

*Proof:* From (5), $B = B_1 \otimes B_2 = \left[\begin{array}{ccc} e_{l_1} & \cdots & e_{l_{pq}} \end{array}\right] \in \mathbb{R}^{nm\times pq}$. Let $\mathcal{K} = \{l_1, \ldots, l_{pq}\}$ be the leader indices of $\mathcal{G}$. Replacing $b_l$ in (8) by $B$, the controllability Gramian of $(\mathcal{L}(\mathcal{G}), B)$ is given by

$$\mathcal{W} = \int_0^\infty e^{-\mathcal{L}\tau} BB^T e^{-\mathcal{L}\tau} d\tau = U\Gamma U^T, \tag{13}$$

where $\Gamma = \int_0^\infty e^{-\Lambda\tau} U^T BB^T U e^{-\Lambda\tau} d\tau$ with $\Gamma \in \mathbb{R}^{mn\times mn}$ and $U \in \mathbb{R}^{mn\times mn}$ as defined in (9). From (11) and (13), the average controllability of $(\mathcal{L}(\mathcal{G}), B)$ can be written as

$$\begin{aligned}\operatorname{tr}(\mathcal{W}) &= \operatorname{tr}(\Gamma) \\ &= \int_0^\infty e^{-2\lambda_1\tau} \sum_{k\in\mathcal{K}} u_{k,1}^2 + \cdots + e^{-2\lambda_{mn}\tau} \sum_{k\in\mathcal{K}} u_{k,mn}^2 d\tau\end{aligned} \tag{14}$$

where $u_{i,j}$ represents the $ij$th entry of $U$. Since $\int_0^\infty e^{-2\lambda_i\tau} d\tau = \frac{1}{2\lambda_i}$, (14) can be further simplified into

$$\operatorname{tr}(\mathcal{W}) = \sum_{k\in\mathcal{K}} \sum_{i=1}^{mn} \frac{1}{2\lambda_i} u_{k,i}^2.$$

Using Lemma 1,

$$\operatorname{tr}(\mathcal{W}) = \sum_{k=1}^{p}\sum_{l=1}^{q}\sum_{i=1}^{n}\sum_{j=1}^{m} \frac{1}{2(\mu_i + \eta_j)} \vartheta_{l_k^1,i}^2 w_{l_l^2,j}^2. \tag{15}$$

where $\vartheta_{l_k^1,:}, (k = 1, 2, \cdots, p)$ and $w_{l_l^2,:}, (l = 1, 2, \cdots, q)$ are the $l_k^1$th and $l_l^2$th rows of the eigenvector matrices of $\mathcal{G}_1$ and $\mathcal{G}_2$, i.e., $\vartheta$ and $w$ respectively, corresponding to the leader node $v_{l_k^1}$ and $v_{l_l^2}$. ■

Following similar discussion, Theorem 2 indicates that, for a Cartesian product composite system with multiple leaders, its average controllability can also be inferred from the eigenvalues of the factor graph Laplacians and the associated rows of the eigenvector matrices corresponding to the leaders.

### *B. Characterizations of Volumetric Control Energy*

This section characterizes the volumetric control energy of the composite system $(\mathcal{L}, b_l)$ based on its factor systems $(\mathcal{L}_1, b_1)$ and $(\mathcal{L}_2, b_2)$.

**Theorem 3.** *Consider two factor systems $(\mathcal{L}(\mathcal{G}_1), b_1)$ and $(\mathcal{L}(\mathcal{G}_2), b_2)$ and its corresponding composite system $(\mathcal{L}(\mathcal{G}_1 \square \mathcal{G}_2), b_l)$ in (6), where $\mathcal{G}_1$ has $n$ nodes and $\mathcal{G}_2$ has $m$ nodes. The volumetric control energy $\log\det\mathcal{W}$ of the composite system can be characterized as*

$$\log\det\mathcal{W} = m\log\det\mathcal{W}_1 + n\log\det\mathcal{W}_2 + c,$$

*where $\mathcal{W}$, $\mathcal{W}_1$, and $\mathcal{W}_2$ are controllability Gramians of the systems $(\mathcal{L}, b_l)$, $(\mathcal{L}_1, b_1)$, and $(\mathcal{L}_2, b_2)$, respectively, and $c = \log\det\overline{\Gamma} - m\log\det\overline{\Gamma}_1 - n\log\det\overline{\Gamma}_2$ is a constant determined by* $\operatorname{eig}(\mathcal{L}_1)$ *and* $\operatorname{eig}(\mathcal{L}_2)$.

*Proof:* From (8), the volumetric control energy of $(\mathcal{L}, b_l)$ can be written as

$$\begin{aligned}\log\det\mathcal{W} &= \log\left(\det U \det\Gamma \det U^T\right) \\ &= \log\det\Gamma,\end{aligned} \tag{16}$$

where $\det U \det U^T = 1$ is used since $U$ is an orthogonal matrix. From (10), $\Gamma$ can be rewritten as $\Gamma = \overline{U}\overline{\Gamma}\overline{U}$, where $\overline{U} = \operatorname{diag}\{u_{l,1}, u_{l,2}, \ldots, u_{l,mn}\}$ and $\overline{\Gamma} = \left[\overline{\Gamma}_{ij}\right] \in \mathbb{R}^{mn\times mn}$ with $\overline{\Gamma}_{ij} = \frac{1}{\lambda_i+\lambda_j}$. Based on $\overline{\Gamma}$, $\overline{U}$, and (16),

$$\begin{aligned}\log\det\mathcal{W} &= \log\left(\det\overline{U}\det\overline{\Gamma}\det\overline{U}\right) \\ &= 2\sum_{i=1}^{mn}\log u_{l,i} + \log\det\overline{\Gamma}.\end{aligned} \tag{17}$$

Expressions, similar to (17), can be obtained for factor systems $(\mathcal{L}_1, b_1)$ and $(\mathcal{L}_2, b_2)$ as

$$\log\det\mathcal{W}_1 = \log\det\left(\overline{V}\overline{\Gamma}_1\overline{V}\right) = 2\sum_{i=1}^{n}\log\vartheta_{l_1,i} + \log\det\overline{\Gamma}_1$$

and

$$\log\det\mathcal{W}_2 = \log\det\left(\overline{W}\overline{\Gamma}_2\overline{W}\right) = 2\sum_{i=1}^{m}\log w_{l_2,i} + \log\det\overline{\Gamma}_2,$$

where $\overline{V} = \operatorname{diag}\{\vartheta_{l_1,1}, \vartheta_{l_1,2}, \ldots, \vartheta_{l_1,n}\}$, $\overline{\Gamma}_1 \in \mathbb{R}^{n\times n}$ with the $ij$th entry

$$\left(\overline{\Gamma}_1\right)_{ij} = \frac{1}{\mu_i + \mu_j}$$

for $i, j \in \{1, \ldots, n\}$, $\overline{W} = \operatorname{diag}\{w_{l_2,1}, w_{l_2,2}, \ldots, w_{l_2,m}\}$, and $\overline{\Gamma}_2 \in \mathbb{R}^{m\times m}$ with the $ij$th entry

$$\left(\overline{\Gamma}_2\right)_{ij} = \frac{1}{\eta_i + \eta_j}$$

for $i, j \in \{1, \ldots, m\}$.

By Lemma 1, $\overline{U} = \overline{V} \otimes \overline{W}$, and the fact that $\det\left(\overline{V}\otimes\overline{W}\right) = \left(\det\overline{V}\right)^m\left(\det\overline{W}\right)^n$, $\log\det\mathcal{W}$ in (17) can be written in terms of $\log\det\mathcal{W}_1$ and $\log\det\mathcal{W}_2$ as

$$\begin{aligned}\log\det\mathcal{W} &= \log\left(\det\left(\overline{V}\otimes\overline{W}\right)\det\overline{\Gamma}\det\left(\overline{V}\otimes\overline{W}\right)\right) \\ &= \log\left(\det\overline{V}\right)^{2m} + \log\left(\det\overline{W}\right)^{2n} + \log\det\overline{\Gamma} \\ &= 2m\sum_{j=1}^{n}\log\vartheta_{l_1,j} + 2n\sum_{k=1}^{m}\log w_{l_2,k} + \log\det\overline{\Gamma} \\ &= m\log\det\mathcal{W}_1 - m\log\det\overline{\Gamma}_1 \\ &\quad + n\log\det\mathcal{W}_2 - n\log\det\overline{\Gamma}_2 + \log\det\overline{\Gamma},\end{aligned}$$

which completes the proof. ■

Theorem 3 indicates that the volumetric control energy of a composite system $(\mathcal{L}, b_l)$ can be inferred from that of its factor systems, i.e., $\log\det\mathcal{W}_1$ and $\log\det\mathcal{W}_2$, and a constant $c$. Note that $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$ are defined based on $\operatorname{eig}(\mathcal{L}_1)$ and $\operatorname{eig}(\mathcal{L}_2)$, respectively, and, by Lemma 1, $\overline{\Gamma}$ also depends on $\operatorname{eig}(\mathcal{L}_1)$ and $\operatorname{eig}(\mathcal{L}_2)$. As a result, $c$ is a constant determined by $\mathcal{L}_1$ and $\mathcal{L}_2$. In addition, since the selection of leader nodes (i.e., design of $b_1$ and $b_2$) does not affect $\mathcal{L}_1$ and $\mathcal{L}_2$, Theorem 3 implies that, if the volumetric control energy of the factor systems is individually maximized then the volumetric control energy of the composite system is maximum. In addition, any change in the volumetric control energy of the composite system can be computed precisely from the change in the volumetric control energy of factor systems. This provides a means for network design, where local factor systems can be individually designed for improved volumetric control energy of the composite system.

**Corollary 1.** *Consider a composite system $(\mathcal{L}(\mathcal{G}_1 \square \mathcal{G}_2), b_l)$ constructed by two factor systems $(\mathcal{L}(\mathcal{G}_1), b_1)$ and $(\mathcal{L}(\mathcal{G}_2), b_2)$. Determining $b_l$ (i.e., selecting a leader node $v_l$ in $\mathcal{G}$) to maximize the volumetric control energy of $(\mathcal{L}, b_l)$ is equivalent to determining $b_1$ and $b_2$ (i.e., selecting the leaders $v_{l_1}$ and $v_{l_2}$ in $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively) such that the volumetric control energy of $(\mathcal{L}_1, b_1)$ and $(\mathcal{L}_2, b_2)$ are individually maximized.*

Theorem 3 can be extended for a multi-leader case. Suppose that $\mathcal{G}_1$ has $p$ leaders and $\mathcal{G}_2$ has $q$ leaders, i.e., $B_1 = \left[\begin{array}{ccc} e_{l_1^1} & \cdots & e_{l_p^1} \end{array}\right] \in \mathbb{R}^{n\times p}$ and $B_2 = \left[\begin{array}{ccc} e_{l_1^2} & \cdots & e_{l_q^2} \end{array}\right] \in \mathbb{R}^{m\times q}$. Let $(\mathcal{L}(\mathcal{G}), B)$ be the composite system formed by the Cartesian product of $(\mathcal{L}(\mathcal{G}_1), B_1)$ and $(\mathcal{L}(\mathcal{G}_2), B_2)$, where $B = B_1 \otimes B_2 = \left[\begin{array}{ccc} e_{l_1} & \cdots & e_{l_{pq}} \end{array}\right] \in \mathbb{R}^{nm\times pq}$. Denote by $\mathcal{K} = \{l_1, \ldots, l_{pq}\}$ the leader indices of $\mathcal{G}$. Following similar analysis as in the proof of Theorem 2, one has

$$\log\det \mathcal{W} = \log\det\left(U\varGamma U^T\right) = \log\det \varGamma,$$

where $\varGamma = \int_0^\infty e^{-\varLambda\tau}U^TBB^TUe^{-\varLambda\tau}d\tau$ with $\varLambda = \operatorname{diag}\{\lambda_1, \ldots, \lambda_{mn}\} \in \mathbb{R}^{mn\times mn}$ and $U = \left[\begin{array}{ccc} u_1 & \cdots & u_{mn} \end{array}\right] \in \mathbb{R}^{mn\times mn}$ being the eigenvalue and eigenvector matrix of $\mathcal{L}(\mathcal{G})$, respectively. Substituting $B$ into (9), the $ij$th entry of $\varGamma$ can be written as

$$\varGamma_{ij} = \frac{\sum_{k\in\mathcal{K}} u_{k,i}u_{k,j}}{\lambda_i + \lambda_j}, . \tag{18}$$

where $u_k$, $k \in \mathcal{K}$, represents the eigenvector of $\mathcal{L}$ corresponding to the leaders. Since any eigenpair $(\lambda_i, u_i)$ in (18) can be replaced by the eigenpairs of $\mathcal{L}_1$ and $\mathcal{L}_2$ using Lemma 1, it is evident that the volumetric control energy $\log\det\mathcal{W}$ of $(\mathcal{L}(\mathcal{G}), B)$ with multiple leaders can be inferred from its factor systems (i.e., the eigenvalues and the eigenvectors corresponding to the leaders of $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively). However, unlike Theorem 3, derivation of volumetric control energy in terms of eigenvalues and eigenvectors of $\mathcal{L}_1$ and $\mathcal{L}_2$ is more involved when considering a multi-leader case. Ongoing research aims to characterize how factor systems individually contribute to the volumetric control energy of the composite system.

## V. Conclusion

Energy-related controllability measures, i.e., average controllability and volumetric control energy, are characterized via Cartesian product in this work. Although the current work provides an energy-related perspective to characterize the performance of complex composite networks, this work is far from complete. For instance, the developed results can be potentially used for network design or leader selection for improved energy efficiency. However, the designed network or selected leaders are not guaranteed to ensure classical network controllability. Future research will continue to address these open problems.

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